



Finite solid circular cylinders subjected to arbitrary surface load. Part I — Analytic solution

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Abstract

This paper presents a general framework for obtaining analytic solutions for finite elastic isotropic solid cylinders subjected to arbitrary surface load. The method of solution uses the displacement function approach to uncouple the equations of equilibrium. The most general solution forms for the two displacement functions for solid cylinders are proposed in terms of infinite series, with z - and θ -dependencies in terms of trigonometric and hyperbolic functions, and with r -dependency in terms of Bessel and modified Bessel functions of the first kind of fractional order. All possible combinations of odd and even dependencies of θ and z are included; and the curved boundary loads are expanded into double Fourier Series expansion, while the end boundary loads are expanded into Fourier–Bessel expansion. It is showed analytically that only one set of the end boundary conditions needs to be satisfied. A system of simultaneous equations for the unknown constants is given independent of the type of the boundary loads. This new approach provides the most general theory for the stress analysis of elastic isotropic solid circular cylinders of finite length. Application of the present solution to the stress analysis for the double-punch test is presented in Part II of this study. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Solid circular cylinders are the most commonly used specimens in various standard tests in engineering application, such as the uniaxial compressive strength test, the triaxial compressive strength test, the Brazilian test, the double-punch test, the block punch index test and the point load strength

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test. In fact, the stress analysis of elastic solid circular cylinders is one of the most fundamental problems in theoretical elasticity and has a rich history in solid mechanics.

Pochhammer (1876) appears to be the first to propose a general analytic solution for an infinite circular cylinder subjected to arbitrary surface loads, the same solution was also derived independently by Chree (1889). A typical example of axisymmetric problems for infinite cylinders is the problem of applying band pressure on the curved surface (Timoshenko and Goodier, 1982; Williams, 1996). If the cylinder is semi-infinite in length, the problem has been considered by Horvay and Mirabal (1958).

For finite-solid circular cylinders subjected to arbitrary loads, Dougall (1914) employed three displacement functions and proposed an approximate approach for the stress analysis. Chree (1889) also included a discussion on the stress analysis of a finite cylinder under surface traction in the last section of his pioneering work, but only some restrictive types of surface traction are discussed. For axisymmetric deformations of finite cylinders, Filon (1902) presented an analytic approach for the stress analysis, and provided the first analytic solution considering the effect of friction between the loading platens and the end surfaces of a solid cylinder on the non-uniform stress distribution within the cylinder under compression. Employing Love (1944) stress function, Saito (1952, 1954) also proposed a general solution form for axisymmetric stress analysis, in terms of Bessel and modified Bessel functions of the first kind of zero and first orders for r -dependency, and in terms of trigonometric and hyperbolic functions in z -dependency. Using a similar solution technique of series expansion, Ogaki and Nakajima (1983) proposed appropriate forms for two stress functions and analyzed the stress field in a solid circular cylinder subjected to parabolically distributed loads on the central part of the end surfaces. By using Saito's (1952) approach, Watanabe (1996) derived an analytic solution for axisymmetric finite cylinders under the uniaxial and confined compression tests, in which the radial displacement at the ends is partially constrained. Actually, the problem of compression test on solid finite cylinders with end friction has been the subject of a number of theoretical studies (e.g. Kimura, 1931; Pickett, 1944; Edelman, 1948; Balla, 1960a, 1960b; Brady, 1971; Peng, 1971; Al-Chalabi and Huang, 1974; Al-Chalabi et al., 1974; Chau, 1997, 1998b). This analysis has been found useful in the interpretation of the strength of rock in uniaxial and triaxial tests (Kotte and Berczes, 1969). For problems with displacements applied on the end surfaces and with zero traction on the curved surface, Robert and Keer (1987a, 1987b) considered the stress singularities at the flat ends of the cylinder. Wei et al. (1999) used the displacement function approach and presented an analytic solution for the axial Point Load Strength Test (PLST), which provides an improvement over the approximation by Wijk (1978). Another type of problems of finite solid cylinders that has been solved analytically is the torsion problem. For example, the twisting of a finite cylinder by a pair of identical annular stamps attached to its ends was considered by Hasegawa (1984), and the twisting of a finite solid cylinder with free curved surface and fixed base by a rigid die attached to the top surface was considered by Gladwell and Lemczyk (1990). Except for the studies by Dougall (1914) and Chree (1889), all of these analyses are only restricted to axisymmetric problems of finite elastic cylinders; while Dougall's (1914) approach is of approximate nature and Chree's (1889) discussion is rather restrictive.

For non-axisymmetric deformations of finite solid cylinders, Wijk (1980) derived a simple approximation for the tensile stress at the center of a cylinder subjected to the diametral PLST, in which two point forces are diametrically applied on the curved surface of the cylinder. Chau (1998a) introduced two displacement functions and derived a solution for a finite circular cylinder under the action of two diametral indentors and with constrained shear displacements on the two end surfaces. This solution, however, is only an approximation for cylinders under the diametral PLST. To model the actual traction free end boundaries (which is the realistic boundary condition for the PLST), Chau and Wei (1999) proposed more general solution forms for the two displacement functions such that all boundary conditions are satisfied exactly. However, there is no closed-form solution for finite elastic circular cylinders under arbitrary loads on the curved and end surfaces.

Therefore, this paper, Part I of the present study, presents a general analytic solution for finite elastic isotropic solid cylinders under arbitrary surface load. The method of solution is generalized from those used by Wei et al. (1999) and Chau and Wei (1999). Complete solution forms for the displacement functions are introduced here such that any traction problems of finite isotropic solid cylinders can be solved exactly. The tractions on the curved surface are expanded into double Fourier series expansion, while the tractions on the end surfaces are expanded into Fourier–Bessel series expansion, in order to match the internal stress field resulting from the general solution forms of the displacement functions. The solutions by Filon (1902), Saito (1952, 1954), Watanabe (1996), Chau (1998a), Chau and Wei (1999) and Wei et al. (1999) can be considered as special cases of the present solution; that is, they can be re-derived independently using the present unified approach. Part II of this study will apply the present solution to the stress analysis of solid cylinders subjected to the double-punch test (Wei and Chau, 1999).

2. Governing equations

Consider a homogeneous and isotropic elastic cylinder of radius R (or diameter D) and length $2L$ in a cylindrical co-ordinate system as shown in Fig. 1. The stress and strain tensors are related by the following Hook's law

$$\sigma_{\alpha\beta} = 2G\varepsilon_{\alpha\beta} + \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} \quad (1)$$

where $\alpha, \beta, \gamma = r, \theta, z$; G and λ are the Lamé constants (G is normally referred as the shear modulus); and repeated indices in Eq. (1) imply summation. The Cauchy stress and strain tensors are denoted by σ

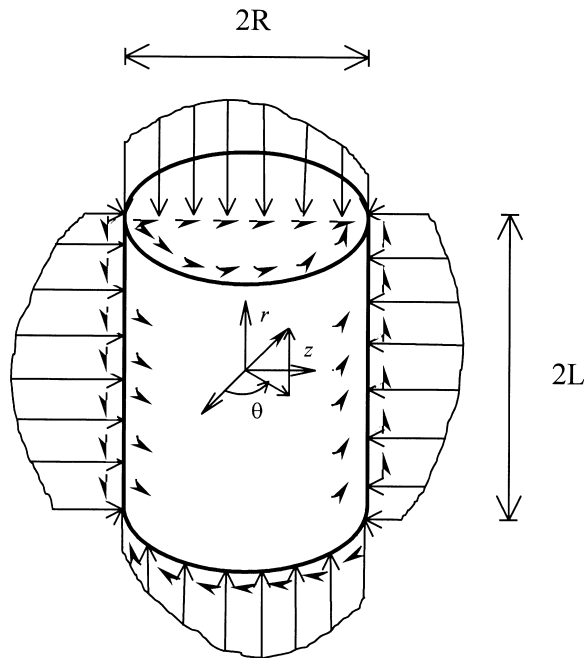


Fig. 1. A sketch of a finite solid circular cylinder of length $2L$ and radius R subjected to arbitrary tractions on the curved and end surfaces.

and $\boldsymbol{\varepsilon}$, respectively. The strain tensor is related to the displacement ($\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z$) by

$$\boldsymbol{\varepsilon} = \frac{1}{2}[(\nabla \mathbf{u})^T + \nabla \mathbf{u}] \quad (2)$$

where

$$\nabla \mathbf{u} = \mathbf{e}_r \frac{\partial \mathbf{u}}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial \mathbf{u}}{\partial \theta} + \mathbf{e}_z \frac{\partial \mathbf{u}}{\partial z} \quad (3)$$

In terms of cylindrical coordinates, the physical components of the strain tensor given in Eq. (2) are

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_{r\theta} = \frac{1}{2} \left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right),$$

$$\varepsilon_{z\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right), \quad \varepsilon_{rz} = \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \quad (4)$$

In the absence of body force, the equations of equilibrium, $\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$, in terms of displacements are (e.g., Malvern, 1969):

$$(\lambda + 2G) \frac{\partial e}{\partial r} + \frac{2G}{r} \frac{\partial \Omega_{r\theta}}{\partial \theta} + 2G \frac{\partial \Omega_{rz}}{\partial z} = 0 \quad (5)$$

$$(\lambda + 2G) \frac{1}{r} \frac{\partial e}{\partial \theta} + 2G \frac{\partial \Omega_{z\theta}}{\partial z} + 2G \frac{\partial \Omega_{\theta r}}{\partial r} = 0 \quad (6)$$

$$(\lambda + 2G) \frac{\partial e}{\partial z} + \frac{2G}{r} \frac{\partial}{\partial r} (r \Omega_{zr}) + \frac{2G}{r} \frac{\partial \Omega_{z\theta}}{\partial \theta} = 0 \quad (7)$$

where $\boldsymbol{\Omega} = [(\nabla \mathbf{u})^T - \nabla \mathbf{u}]/2$ and $e = \nabla \cdot \mathbf{u}$ are the spin tensor and the volumetric strain, respectively.

When the cylinder is subjected to arbitrary traction on its surface, the general boundary conditions are

$$\sigma_{rr} = f_{1r}(z, \theta), \quad \sigma_{rz} = f_{1z}(z, \theta), \quad \sigma_{r\theta} = f_{1\theta}(z, \theta) \quad \text{on } r = R \quad (8)$$

$$\sigma_{zr} = f_{2r}(r, \theta), \quad \sigma_{zz} = f_{2z}(r, \theta), \quad \sigma_{z\theta} = f_{2\theta}(r, \theta) \quad \text{on } z = +L \quad (9)$$

$$\sigma_{zr} = f_{3r}(r, \theta), \quad \sigma_{zz} = f_{3z}(r, \theta), \quad \sigma_{z\theta} = f_{3\theta}(r, \theta) \quad \text{on } z = -L \quad (10)$$

where f_{ij} ($i = 1, 2, 3$; $j = r, z, \theta$) are the prescribed tractions on the curved surface and on the end surfaces of the cylinder. The first subscript $i = 1, 2, 3$ indicates the curved, top and bottom end surfaces, respectively; the second subscript $j (= r, \theta, z)$ indicates the direction along which the traction acts. To simplify the later discussion, these boundary conditions are called $BCij$ ($i, j = 1, 2, 3$) as defined in Table 1. For example, the first part of Eq. (8) is denoted by BC11.

3. Method of solution

The main objective of this paper is to obtain the exact solution satisfying both the equations of equilibrium (5)–(7) and the boundary conditions (8)–(10). Similar to the analyses by Chau (1998a, 1998b) and Chau and Wei (1999), two displacement functions Φ and Ψ are introduced

$$u_r = \frac{\partial^2 \Phi}{\partial r \partial z} + \frac{1}{r} \frac{\partial \Psi}{\partial \theta}, \quad u_\theta = \frac{1}{r} \frac{\partial^2 \Phi}{\partial \theta \partial z} - \frac{\partial \Psi}{\partial r}, \quad u_z = - \left[2(1 - \nu) \nabla_1 \Phi + (1 - 2\nu) \frac{\partial^2 \Phi}{\partial z^2} \right] \tag{11}$$

where

$$\nabla_1 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \tag{12}$$

Substitution of Eq. (11) into Eqs. (5)–(7) yields two uncoupled governing equations for the displacement functions Φ and Ψ :

$$\nabla^4 \Phi = \nabla^2 \nabla^2 \Phi = 0, \quad \nabla^2 \Psi = 0 \tag{13}$$

where ∇^2 is the Laplacian operator, or $\nabla^2 = \nabla_1 + \partial^2 / \partial z^2$. That is, Φ and Ψ satisfy the biharmonic and harmonic equations, respectively.

In terms of these two displacement functions, the physical components of the stress tensor can be obtained by substituting Eq. (11) into Eqs. (4) and (1) as,

$$\sigma_{rr} = -2\nu G \nabla^2 \frac{\partial \Phi}{\partial z} + 2G \left[\frac{\partial^3 \Phi}{\partial z \partial r^2} + \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial \theta} \right) \right] \tag{14}$$

$$\sigma_{\theta\theta} = -2\nu G \nabla^2 \frac{\partial \Phi}{\partial z} + 2G \left[\frac{1}{r} \frac{\partial^2 \Phi}{\partial z \partial r} + \frac{1}{r^2} \frac{\partial^3 \Phi}{\partial \theta^2 \partial z} - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial \theta} \right) \right] \tag{15}$$

$$\sigma_{zz} = -2G \left[(2 - \nu) \frac{\partial}{\partial z} \nabla^2 - \frac{\partial^3}{\partial z^3} \right] \Phi \tag{16}$$

$$\sigma_{rz} = 2G \left[- (1 - \nu) \frac{\partial}{\partial r} \nabla_1 + \nu \frac{\partial^3}{\partial r \partial z^2} \right] \Phi + \frac{G}{r} \frac{\partial^2 \Psi}{\partial \theta \partial z} \tag{17}$$

Table 1
The definitions for boundary conditions BC_{ij} ($i, j = 1, 2, 3$)

| Surface | Direction | | |
|----------------|------------------------|------------------------|----------------------------------|
| | 1 (r) | 2 (z) | 3 (θ) |
| 1 ($r = R$) | $\sigma_{rr} = f_{1r}$ | $\sigma_{rz} = f_{1z}$ | $\sigma_{r\theta} = f_{1\theta}$ |
| 2 ($z = L$) | $\sigma_{zr} = f_{2r}$ | $\sigma_{zz} = f_{2z}$ | $\sigma_{z\theta} = f_{2\theta}$ |
| 3 ($z = -L$) | $\sigma_{zr} = f_{3r}$ | $\sigma_{zz} = f_{3z}$ | $\sigma_{z\theta} = f_{3\theta}$ |

$$\sigma_{z\theta} = 2G \left[- (1 - \nu) \frac{1}{r} \frac{\partial}{\partial \theta} \nabla_1 + \nu \frac{1}{r} \frac{\partial^3}{\partial \theta \partial z^2} \right] \Phi - G \frac{\partial^2 \Psi}{\partial r \partial z} \quad (18)$$

$$\sigma_{r\theta} = 2G \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial^2 \Phi}{\partial \theta \partial z} \right) + \frac{1}{2} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} - \frac{\partial^2 \Psi}{\partial r^2} \right) \right] \quad (19)$$

4. Series expressions for the two displacement functions

The most difficult step to solve the problem is to find the appropriate and complete solution forms for the two displacement potentials Φ and Ψ , which should satisfy both of the governing equations given in Eq. (13), and the boundary conditions (8)–(10) for any arbitrary applied tractions f_{ij} .

The method of separation of variables is employed here to solve Eq. (13). In particular, we assume the series solution for Ψ as

$$\Psi(r, z, \theta) = \sum_{n=0}^{\infty} [\psi_1(r, z) \cos(\omega_n \theta) + \psi_2(r, z) \sin(\omega_n \theta)] \quad (20)$$

where ω_n is defined as $\omega_n = 2n\pi/T$, T is the periodicity of Ψ in θ , which should match the periodicity of the external applied traction in θ . Substitution of Eq. (20) into Eq. (13) leads to the governing equation for function $\psi_i(r, z)$, $i = 1, 2$,

$$\frac{\partial^2 \psi_i}{\partial r^2} + \frac{1}{r} \frac{\partial \psi_i}{\partial r} + \frac{\partial^2 \psi_i}{\partial z^2} - \frac{\omega_n^2}{r^2} \psi_i = 0 \quad (21)$$

The general solution of Eq. (21) is

$$\psi_i(r, z) = [AI_{\omega_n}(\eta r) + BK_{\omega_n}(\eta r)] \frac{\sin(\eta z)}{\cos(\eta z)} + [CJ_{\omega_n}(\gamma r) + DY_{\omega_n}(\gamma r)] \frac{\sinh(\gamma z)}{\cosh(\gamma z)} \quad (22)$$

where $J_{\omega_n}(\gamma r)$, $Y_{\omega_n}(\gamma r)$, $I_{\omega_n}(\eta r)$ and $K_{\omega_n}(\eta r)$ are Bessel functions and modified Bessel functions of the first and second kinds with fractional order ω_n . Parameters A , B , C , D , η and γ are constants to be determined. Because the stress field at the center of the finite solid cylinder must be finite, all terms that relate to $Y_{\omega_n}(\gamma r)$ and $K_{\omega_n}(\eta r)$ must be discarded. Consequently, the general expression for Ψ is assumed as:

$$\Psi = -\frac{1}{2G} \left\{ E_{00} r^{\omega_n} + \sum_{n=0}^{\infty} \left\{ E_{0n}^{(l)} r^{\omega_n} + \sum_{m=1}^{\infty} \frac{E_{nm}^{(k)}}{\eta_m^2} I_{\omega_n}(\eta_m r) \frac{\cos(\eta_m z)}{\sin(\eta_m z)} + \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)}}{\gamma_s^2} J_{\omega_n}(\gamma_s r) \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right\} \frac{\sin(\omega_n \theta)}{\cos(\omega_n \theta)} \right\} \quad (23)$$

where G is the shear modulus, $\eta_m = m\pi/L$, $\gamma_s = \lambda_s/R$, λ_s is the s th root of $J'_{\omega_n}(x) = 0$. The characteristics of λ_s will be discussed in later section. $E_{00} E_{0n}^{(l)}$, $E_{nm}^{(k)}$ and $F_{sn}^{(k)}$ ($l = 1, 2$; $k = 1, 2, 3, 4$) are unknown constants to be determined by the boundary conditions. The superscript l range from 1 to 2 since the θ -dependency can be either sin or cos; and for general cases, we should include both $l = 1$ and 2. For the superscript k , we can have four combinations for the z - and θ -dependencies; and in general all four combinations are needed for the most general case of applied traction. The corresponding z - and θ -

dependencies attached to the unknown constants for each k and l are tabulated in Table 2. The summation for m in Eq. (23) can be used to fit any z -dependency of applied traction on the curved surface while the summation for s can take care of any r -dependency of applied traction on the end surfaces. Note that the term $\psi_i(r, z) = Er^{\omega_n}$ is resulted by considering the special case of $\eta_m = 0$; and that the term for E_{00} will lead to constant shear stress field for σ_{rz} only.

To find the general solution for Φ , we let $\nabla^2\Phi = \bar{\Psi}$ and note that the general solution for $\bar{\Psi}$ is the same as those for Ψ given in Eq. (22). By back substitution of this solution into $\nabla^2\Phi = \bar{\Psi}$ and by careful inspection, the following general solution for Φ is obtained:

$$\Phi = -\frac{1}{2G} \left\{ A_{00} \frac{z^3}{6} + C_{00} \frac{z}{2} r^2 + \sum_{n=0}^{\infty} \left\{ A_{0n}^{(l)} r^{\omega_n} z + \sum_{m=1}^{\infty} \frac{1}{\eta_m^3} \left[A_{mn}^{(k)} r \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} + B_{mn}^{(k)} I_{\omega_n}(\eta_m r) \right] \frac{\sin(\eta_m z)}{\cos(\eta_m z)} \right. \right. \\ \left. \left. + \sum_{s=1}^{\infty} \frac{1}{\gamma_s^3} \left[C_{sn}^{(k)} \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} + D_{sn}^{(k)} \gamma_s z \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right] J_{\omega_n}(\gamma_s r) \right\} \begin{matrix} \cos(\omega_n \theta) \\ \sin(\omega_n \theta) \end{matrix} \right\} \quad (24)$$

where A_{00} , C_{00} , $A_{0n}^{(l)}$, $A_{mn}^{(k)}$, $B_{mn}^{(k)}$, $C_{sn}^{(k)}$, $D_{sn}^{(k)}$ ($l = 1, 2$; $k = 1, 2, 3, 4$) are unknown coefficients to be determined by the boundary conditions. Each combination of l and k corresponds to a particular combination of the upper and lower functions of z and θ given in Eq. (24). More specifically, the corresponding z - and θ -dependencies attaching to the unknown constants for each k and l are given in Table 2; and in general, all combinations of k and l are needed for general loading cases. Note that the choices for l given in Table 2 will lead to $\sigma_{rr}(r, z, \theta)$ being an even function in θ [i.e. $\cos(\omega_n \theta)$] if $l = 1$ and being an odd function [i.e. $\sin(\omega_n \theta)$] if $l = 2$. In addition, the z -dependency/ θ -dependency for $\sigma_{rr}(r, z, \theta)$ will be even/even, even/odd, odd/even and odd/odd for $k = 1, 2, 3, 4$, respectively. Note that

Table 2
The θ - and z -dependencies for superscripts l ($= 1, 2$) and k ($= 1, 2, 3, 4$) used for the displacement functions Ψ and Φ given in Eqs. (23) and (24)

| Constants for Ψ and Φ | Superscripts l/k | θ -dependency | z -dependency |
|---------------------------------|--------------------|----------------------|-----------------|
| $E_{0n}^{(l)} / A_{0n}^{(l)}$ | 1 | sin/cos | – |
| | 2 | cos/sin | – |
| $E_{mn}^{(k)} / F_{sn}^{(k)}$ | 1 | sin | cos/cosh |
| | 2 | cos | cos/cosh |
| | 3 | sin | sin/sinh |
| | 4 | cos | sin/sinh |
| $A_{mn}^{(k)} / B_{mn}^{(k)}$ | 1 | cos | sin |
| | 2 | sin | sin |
| | 3 | cos | cos |
| | 4 | sin | cos |
| $C_{sn}^{(k)} / D_{sn}^{(k)}$ | 1 | cos | sinh/zcosh |
| | 2 | sin | sinh/zcosh |
| | 3 | cos | cosh/zsinh |
| | 4 | sin | cosh/zsinh |

the constants A_{00} and C_{00} will only lead to constant normal stresses. In addition, it is straightforward to show that Eqs. (23) and (24) satisfy the governing equations given in Eq. (13) identically.

5. General expressions for stresses

Substitution of Eqs. (23) and (24) into Eqs. (14)–(19) leads to the following general expressions for the stress components:

$$\begin{aligned}
 \sigma_{rr} = & (2\nu - 1)C_{00} + \nu A_{00} + \sum_{n=0}^{\infty} \left\{ -\omega_n(\omega_n - 1) \left(A_{0n}^{(l)} \frac{\cos(\omega_n \theta)}{\sin(\omega_n \theta)} + E_{0n}^{(l)} \frac{\cos(\omega_n \theta)}{-\sin(\omega_n \theta)} \right) r^{\omega_n - 2} \right. \\
 & + \left\{ \sum_{m=1}^{\infty} \left[A_{mn}^{(k)} \left[2\nu I_{\omega_n}(\eta_m r) - \frac{1}{\eta_m^2} \frac{\partial^2}{\partial r^2} \left(r \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} \right) \right] - \frac{B_{mn}^{(k)}}{\eta_m^2} \frac{\partial^2 I_{\omega_n}(\eta_m r)}{\partial r^2} \right] \cos(\eta_m z) \right. \\
 & + \sum_{s=1}^{\infty} \left[\left(C_{sn}^{(k)} + D_{sn}^{(k)} \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} + D_{sn}^{(k)} \gamma_s z \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} \right) \left[\frac{1}{\gamma_s^2 r} \frac{\partial J_{\omega_n}(\gamma_s r)}{\partial r} - \frac{\omega_n^2}{\gamma_s^2 r^2} J_{\omega_n}(\gamma_s r) \right] \right. \\
 & + \left. \left. \left[\left(C_{sn}^{(k)} + (2\nu + 1)D_{sn}^{(k)} \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} + D_{sn}^{(k)} \gamma_s z \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} \right) J_{\omega_n}(\gamma_s r) \right] \right\} \cos(\omega_n \theta) \right. \\
 & \left. - \left\{ \sum_{m=1}^{\infty} \frac{E_{mn}^{(k)} \omega_n}{\eta_m^2} \frac{\partial}{\partial r} \left(\frac{I_{\omega_n}(\eta_m r)}{r} \right) \frac{\cos(\eta_m z)}{\sin(\eta_m z)} + \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)} \omega_n}{\gamma_s^2} \frac{\partial}{\partial r} \left(\frac{J_{\omega_n}(\gamma_s r)}{r} \right) \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right\} \cos(\omega_n \theta) \right. \\
 & \left. - \left\{ \sum_{m=1}^{\infty} \frac{E_{mn}^{(k)} \omega_n}{\eta_m^2} \frac{\partial}{\partial r} \left(\frac{I_{\omega_n}(\eta_m r)}{r} \right) \frac{\cos(\eta_m z)}{\sin(\eta_m z)} + \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)} \omega_n}{\gamma_s^2} \frac{\partial}{\partial r} \left(\frac{J_{\omega_n}(\gamma_s r)}{r} \right) \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right\} -\sin(\omega_n \theta) \right\} \quad (25)
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{\theta\theta} = & (2\nu - 1)C_{00} + \nu A_{00} + \sum_{n=0}^{\infty} \left\{ \omega_n(\omega_n - 1) \left(A_{0n}^{(l)} \frac{\cos(\omega_n \theta)}{\sin(\omega_n \theta)} + E_{0n}^{(l)} \frac{\cos(\omega_n \theta)}{-\sin(\omega_n \theta)} \right) r^{\omega_n - 2} \right. \\
 & + \left\{ \sum_{m=1}^{\infty} \left[A_{mn}^{(k)} \left[2\nu I_{\omega_n}(\eta_m r) - \frac{1}{\eta_m^2} \frac{\partial}{\partial r} \left(r \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} \right) + \frac{\omega_n^2}{\eta_m^2 r} \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} \right] \right. \\
 & - \frac{B_{mn}^{(k)}}{\eta_m^2} \left[\frac{1}{r} \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} - \frac{\omega_n^2}{r^2} J_{\omega_n}(\eta_m r) \right] \left. \right\} \cos(\eta_m z) + \sum_{s=1}^{\infty} \left\{ 2\nu D_{sn}^{(k)} J_{\omega_n}(\gamma_s r) \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right. \\
 & - \left. \left[\left(C_{sn}^{(k)} + D_{sn}^{(k)} \right) \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} + D_{sn}^{(k)} \gamma_s z \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} \right] \left[\frac{1}{\gamma_s^2 r} \frac{\partial J_{\omega_n}(\gamma_s r)}{\partial r} - \frac{\omega_n^2}{\gamma_s^2 r^2} J_{\omega_n}(\gamma_s r) \right] \right\} \cos(\omega_n \theta) \\
 & + \left\{ \sum_{m=1}^{\infty} \frac{E_{mn}^{(k)} \omega_n}{\eta_m^2} \frac{\partial}{\partial r} \left(\frac{I_{\omega_n}(\eta_m r)}{r} \right) \frac{\cos(\eta_m z)}{\sin(\eta_m z)} + \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)} \omega_n}{\gamma_s^2} \frac{\partial}{\partial r} \left(\frac{J_{\omega_n}(\gamma_s r)}{r} \right) \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right\} \cos(\omega_n \theta) \\
 & - \left\{ \sum_{m=1}^{\infty} \frac{E_{mn}^{(k)} \omega_n}{\eta_m^2} \frac{\partial}{\partial r} \left(\frac{I_{\omega_n}(\eta_m r)}{r} \right) \frac{\cos(\eta_m z)}{\sin(\eta_m z)} + \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)} \omega_n}{\gamma_s^2} \frac{\partial}{\partial r} \left(\frac{J_{\omega_n}(\gamma_s r)}{r} \right) \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right\} -\sin(\omega_n \theta) \quad (26)
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{zz} = & 2(2 - \nu)C_{00} + (1 - \nu)A_{00} + \sum_{n=0}^{\infty} \left\{ \left[\sum_{m=1}^{\infty} \left\{ A_{nm}^{(k)} \left[2(2 - \nu)I_{\omega_n}(\eta_m r) + r \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} \right] \right. \right. \right. \\
 & + \left. \left. B_{mn}^{(k)} I_{\omega_n}(\eta_m r) \right\} \cos(\eta_m z) - \sin(\eta_m z) - \sum_{s=1}^{\infty} \left[\left(C_{sn}^{(k)} + (2\nu - 1)D_{sn}^{(k)} \right) \cosh(\gamma_s z) \right. \right. \\
 & \left. \left. + D_{sn}^{(k)} \gamma_s z \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} \right] J_{\omega_n}(\gamma_s r) \right\} \left. \begin{matrix} \cos(\omega_n \theta) \\ \sin(\omega_n \theta) \end{matrix} \right\} \tag{27}
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{rz} = & 9(1 - \nu)E_{00} + \sum_{n=0}^{\infty} \left\{ \left[\sum_{m=1}^{\infty} \left\{ \frac{A_{nm}^{(k)}}{\eta_m} \left[2(1 - \nu) \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} + \frac{\partial}{\partial r} \left(r \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} \right) \right] \right. \right. \right. \\
 & + \left. \left. \frac{B_{mn}^{(k)} \partial I_{\omega_n}(\eta_m r)}{\eta_m \partial r} \right\} \sin(\eta_m z) - \cos(\eta_m z) - \sum_{s=1}^{\infty} \frac{1}{\gamma_s} \left[\left(C_{sn}^{(k)} + 2\nu D_{sn}^{(k)} \right) \sinh(\gamma_s z) \right. \right. \\
 & \left. \left. + D_{sn}^{(k)} \gamma_s z \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right] \frac{\partial J_{\omega_n}(\gamma_s r)}{\partial r} \right\} \left. \begin{matrix} \cos(\omega_n \theta) \\ \sin(\omega_n \theta) \end{matrix} \right\} - \left\{ \sum_{m=1}^{\infty} \frac{E_{mn}^{(k)} \omega_n}{2\eta_m r} I_{\omega_n}(\eta_m r) \frac{-\sin(\eta_m z)}{\cos(\eta_m z)} \right. \\
 & \left. + \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)} \omega_n}{2\gamma_s r} J_{\omega_n}(\gamma_s r) \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} \right\} \left. \begin{matrix} \cos(\omega_n \theta) \\ -\sin(\omega_n \theta) \end{matrix} \right\} \tag{28}
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{z\theta} = & \sum_{n=0}^{\infty} \left\{ \left[\sum_{m=1}^{\infty} \left\{ \frac{\omega_n A_{mn}^{(k)}}{\eta_m} \left[2(1 - \nu) \frac{I_{\omega_n}(\eta_m r)}{r} + \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} \right] + \frac{\omega_n B_{mn}^{(k)} I_{2n}(\eta_m r)}{\eta_m r} \right\} \sin(\eta_m z) \right. \right. \\
 & \left. \left. - \sum_{s=1}^{\infty} \left\{ \frac{\omega_n}{\gamma_s} \left[\left(C_{sn}^{(k)} + 2\nu D_{sn}^{(k)} \right) \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} + D_{sn}^{(k)} \gamma_s z \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right] \frac{J_{\omega_n}(\gamma_s r)}{r} \right\} \right\} \left. \begin{matrix} -\sin(\omega_n \theta) \\ \cos(\omega_n \theta) \end{matrix} \right\} \\
 & + \left\{ \sum_{m=1}^{\infty} \frac{E_{mn}^{(k)} \partial I_{\omega_n}(\eta_m r)}{2\eta_m \partial r} \frac{-\sin(\eta_m z)}{\cos(\eta_m z)} + \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)} \partial J_{\omega_n}(\gamma_s r)}{2\gamma_s \partial r} \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} \right\} \left. \begin{matrix} \sin(\omega_n \theta) \\ \cos(\omega_n \theta) \end{matrix} \right\} \tag{29}
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{r\theta} = & \sum_{n=0}^{\infty} \left\{ -\omega_n(\omega_n - 1) \left(A_{0n}^{(l)} \frac{-\sin(\omega_n\theta)}{\cos(\omega_n\theta)} - E_{0n}^{(l)} \frac{-\sin(\omega_n\theta)}{\cos(\omega_n\theta)} \right) r^{\omega_n-2} - \left\{ \sum_{m=1}^{\infty} \left[\frac{\omega_n A_{mn}^{(k)}}{\eta_m^2} \frac{\partial^2 I_{\omega_n}(\eta_m r)}{\partial r^2} \right. \right. \right. \\
 & + \left. \left. \frac{\omega_n B_{mn}^{(k)}}{\eta_m^2} \left[\frac{1}{r} \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} - \frac{1}{r^2} I_{\omega_n}(\eta_m r) \right] \right\} \frac{\cos(\eta_m z)}{-\sin(\eta_m z)} + \sum_{s=1}^{\infty} \left\{ \frac{\omega_n}{\gamma_s^2} \left[(C_{sn}^{(k)} + D_{sn}^{(k)}) \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right. \right. \\
 & \left. \left. + D_{sn}^{(k)} \gamma_s z \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} \right] \left[\frac{1}{r} \frac{\partial J_{\omega_n}(\gamma_s r)}{\partial r} - \frac{1}{r^2} J_{\omega_n}(\gamma_s r) \right] \right\} \frac{-\sin(\omega_n\theta)}{\cos(\omega_n\theta)} - \left\{ \sum_{m=1}^{\infty} \frac{E_{mn}^{(k)}}{2\eta_m^2} \left[\frac{1}{r} \frac{\partial I_{\omega_n}(\eta_m r)}{\partial r} \right. \right. \\
 & - \left. \left. \frac{\omega_n^2}{r^2} I_{\omega_n}(\eta_m r) - \frac{\partial^2 I_{\omega_n}(\eta_m r)}{\partial r^2} \right] \frac{\cos(\eta_m z)}{\sin(\eta_m z)} + \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)}}{2\gamma_s^2} \left[\frac{1}{r} \frac{\partial J_{\omega_n}(\gamma_s r)}{\partial r} - \frac{\omega_n^2}{r^2} J_{\omega_n}(\gamma_s r) \right. \right. \\
 & \left. \left. - \frac{\partial^2 J_{\omega_n}(\gamma_s r)}{\partial r^2} \right] \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \frac{\sin(\omega_n\theta)}{\cos(\omega_n\theta)} \right\} \tag{30}
 \end{aligned}$$

For the sake of completeness, the proper choice for the upper and lower functions in z and θ corresponding to each value of k ($= 1, 2, 3, 4$) and l ($= 1, 2$) is shown in Table 3. It should be emphasized again that all terms of k and l are needed in general.

6. Determination of unknown coefficients

By using double Fourier expansion technique, all boundary tractions acting on the curved surface of the cylinder given in Eq. (8) can be expressed as (Brown and Churchill, 1993):

$$\begin{aligned}
 \sigma_{rz} = & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \lambda_{mn} \left[a_{mn}^{(\alpha)} \cos(\eta_m z) \cos(\omega_n \theta) + b_{mn}^{(\alpha)} \sin(\eta_m z) \cos(\omega_n \theta) + c_{mn}^{(\alpha)} \cos(\eta_m z) \sin(\omega_n \theta) \right. \right. \\
 & \left. \left. + d_{mn}^{(\alpha)} \sin(\eta_m z) \sin(\omega_n \theta) \right] \right\} \tag{31}
 \end{aligned}$$

Table 3
The θ - and z -dependencies in Eqs. (25)–(30) used for the stresses for superscripts l ($= 1, 2$) and k ($= 1, 2, 3, 4$)^a

| l/k | Dependency | σ_{rr} | σ_{rz} | $\sigma_{r\theta}$ | σ_{zz} | $\sigma_{z\theta}$ |
|---------|------------|---------------|---------------|--------------------|---------------|--------------------|
| $l = 1$ | θ | E1 | E1 | O1 | E1 | O1 |
| $l = 2$ | θ | O1 | O1 | E1 | O1 | E1 |
| $k = 1$ | θ | E1 | E1 | O1 | E1 | O1 |
| | z | E2 | O2 | E2 | E2 | O2 |
| $k = 2$ | θ | O1 | O1 | E1 | O1 | E1 |
| | z | E2 | O2 | E2 | E2 | O2 |
| $k = 3$ | θ | E1 | E1 | O1 | E1 | O1 |
| | z | O2 | E2 | O2 | O2 | E2 |
| $k = 4$ | θ | O1 | O1 | E1 | O1 | E1 |
| | z | O2 | E2 | O2 | O2 | E2 |

^a Symbols: E1 = cos, E2 = cos, cosh, zsinh, O1 = sin, O2 = sin, sinh, zcosh.

where $\alpha = r, z, \theta$, and

$$\lambda_{mn} = \begin{cases} 1/4 & \text{for } m = n = 0 \\ 1/2 & \text{for } n = 0, m > 0 \text{ or } m = 0, n > 0 \\ 1 & \text{for } m > 0, n > 0 \end{cases} \tag{32}$$

$$a_{mn}^{(z)} = \frac{2}{LT} \int_{-T/2}^{T/2} \int_{-L}^L f_{1\alpha}(z, \theta) \cos(\eta_m z) \cos(\omega_n \theta) \, dz \, d\theta,$$

$$b_{mn}^{(z)} = \frac{2}{LT} \int_{-T/2}^{T/2} \int_{-L}^L f_{1\alpha}(z, \theta) \sin(\eta_m z) \cos(\omega_n \theta) \, dz \, d\theta, \tag{33}$$

$$c_{mn}^{(z)} = \frac{2}{LT} \int_{-T/2}^{T/2} \int_{-L}^L f_{1\alpha}(z, \theta) \cos(\eta_m z) \sin(\omega_n \theta) \, dz \, d\theta,$$

$$d_{mn}^{(z)} = \frac{2}{LT} \int_{-T/2}^{T/2} \int_{-L}^L f_{1\alpha}(z, \theta) \sin(\eta_m z) \sin(\omega_n \theta) \, dz \, d\theta. \tag{34}$$

In order to match our general expressions of the stress field with the applied stress given in Eq. (31) on the curved boundary (corresponding to BC11, BC12, and BC13), we first substitute $r = R$ into Eqs. (25), (28) and (30), then expand all functions of z in terms of Fourier sine and cosine series. Finally, the shear and normal stresses on the curved boundary can be expressed as:

$$\begin{aligned} \sigma_{rr} = & (2\nu - 1)C_{00} + \nu A_{00} + \sum_{n=0}^{\infty} \left\{ -\omega_n(\omega_n - 1) \left(A_{0n}^{(l)} \frac{\cos(\omega_n \theta)}{\sin(\omega_n \theta)} + E_{0n}^{(l)} \frac{\cos(\omega_n \theta)}{-\sin(\omega_n \theta)} \right) R^{\omega_n - 2} \right. \\ & - \mathfrak{R}_{0n}^{(l)}(R) \frac{\cos(\omega_n \theta)}{\sin(\omega_n \theta)} + \mathfrak{I}_{0n}^{(l)}(R) \frac{\cos(\omega_n \theta)}{-\sin(\omega_n \theta)} + \sum_{m=1}^{\infty} \left\{ A_{mn}^{(k)} \left[2\nu I_{\omega_n}(\eta_m R) - \frac{2}{\eta_m^2} \frac{\partial^2 I_{\omega_n}(\eta_m R)}{\partial r^2} \right. \right. \\ & \left. \left. - \frac{R}{\eta_m^2} \frac{\partial^3 I_{\omega_n}(\eta_m R)}{\partial r^3} \right] - \frac{B_{mn}^{(k)}}{\eta_m^2} \frac{\partial^2 I_{\omega_n}(\eta_m R)}{\partial r^2} \right\} \cos(\eta_m z) - \sum_{s=1}^{\infty} \left\{ \frac{\omega_n^2}{\gamma_s^2 R^2} \left[(C_{sn}^{(k)} + D_{sn}^{(k)}) \Gamma_{sm}^{(i)} \right. \right. \\ & \left. \left. + D_{sn}^{(k)} A_{sm}^{(i)} \right] - \left[(C_{sn}^{(k)} + (2\nu + 1)D_{sn}^{(k)}) \Gamma_{sm}^{(i)} + D_{sn}^{(k)} A_{sm}^{(i)} \right] \right\} J_{\omega_n}(\gamma_s R) \frac{\cos(\eta_m z)}{\sin(\eta_m z)} \left. \begin{matrix} \cos(\omega_n \theta) \\ \sin(\omega_n \theta) \end{matrix} \right\} \\ & - \left\{ \frac{E_{mn}^{(k)} \omega_n}{\eta_m^2} \left(\frac{1}{R} \frac{\partial I_{\omega_n}(\eta_m R)}{\partial r} - \frac{I_{\omega_n}(\eta_m R)}{R^2} \right) \cos(\eta_m z) \right. \\ & \left. - \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)} \omega_n}{\gamma_s^2} \Gamma_{sm}^{(i)} \frac{J_{\omega_n}(\gamma_s R)}{R^2} \frac{\cos(\eta_m z)}{\sin(\eta_m z)} \right\} \begin{matrix} \cos(\omega_n \theta) \\ -\sin(\omega_n \theta) \end{matrix} \left. \right\} \tag{35} \end{aligned}$$

$$\begin{aligned} \sigma_{rz} = & 9(1-\nu)E_{00} + \sum_{n=0}^{\infty} \left\{ -\mathfrak{N}_{0n}^{(l)}(R) \frac{\cos(\omega_n \theta)}{\sin(\omega_n \theta)} + \sum_{m=1}^{\infty} \left\{ \left[\frac{A_{mn}^{(k)}}{\eta_m} \left[(3-2\nu) \frac{\partial I_{\omega_n}(\eta_m R)}{\partial r} \right. \right. \right. \right. \\ & \left. \left. \left. + R \frac{\partial^2 I_{\omega_n}(\eta_m R)}{\partial r^2} \right] + \frac{B_{mn}^{(k)}}{\eta_m} \frac{\partial I_{\omega_n}(\eta_m R)}{\partial r} \right\} \frac{\sin(\eta_m z)}{\cos(\eta_m z)} \frac{\cos(\omega_n \theta)}{\sin(\omega_n \theta)} - \left\{ \frac{E_{mn}^{(k)} \omega_n}{2 \eta_m R} I_{\omega_n}(\eta_m R) \frac{-\sin(\eta_m z)}{\cos(\eta_m z)} \right. \right. \\ & \left. \left. + \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)} \omega_n}{2 \gamma_s R} J_{\omega_n}(\gamma_s R) \Gamma_{sm}^{(i)} \frac{\sin(\eta_m z)}{\cos(\eta_m z)} \right\} \frac{\cos(\omega_n \theta)}{-\sin(\omega_n \theta)} \right\} \end{aligned} \tag{36}$$

$$\begin{aligned} \sigma_{r\theta} = & \sum_{n=0}^{\infty} \left\{ -\omega_n(\omega_n - 1) \left(\begin{matrix} A_{0n}^{(l)} & -\sin(\omega_n \theta) \\ & \cos(\omega_n \theta) \end{matrix} - E_{0n}^{(l)} \begin{matrix} -\sin(\omega_n \theta) \\ -\cos(\omega_n \theta) \end{matrix} \right) R^{\omega_n - 2} + \mathfrak{P}_{0n}^{(l)}(R) \frac{-\sin(\omega_n \theta)}{\cos(\omega_n \theta)} \right. \\ & \left. + \mathfrak{A}_{0n}^{(l)}(R) \frac{\sin(\omega_n \theta)}{\cos(\omega_n \theta)} - \sum_{m=1}^{\infty} \left\{ \left[\left[\frac{\omega_n A_{mn}^{(k)}}{\eta_m^2} \frac{\partial^2 I_{\omega_n}(\eta_m R)}{\partial r^2} + \frac{\omega_n B_{mn}^{(k)}}{\eta_m^2} \left[\frac{1}{R} \frac{\partial I_{\omega_n}(\eta_m R)}{\partial r} - \frac{1}{R^2} I_{\omega_n}(\eta_m R) \right] \right] \right. \right. \right. \\ & \left. \left. \left. \cos(\eta_m z) - \sum_{s=1}^{\infty} \left\{ \frac{\omega_n}{\gamma_s^2 R^2} \left[\left(C_{sn}^{(k)} + D_{sn}^{(k)} \right) \Gamma_{sm}^{(i)} + D_{sn}^{(k)} A_{sm}^{(i)} \right] J_{\omega_n}(\gamma_s R) \frac{\cos(\eta_m z)}{\sin(\eta_m z)} \right\} \right\} \frac{-\sin(\omega_n \theta)}{\cos(\omega_n \theta)} \right. \right. \\ & \left. \left. + \left\{ \frac{E_{mn}^{(k)}}{2 \eta_m^2} \left[\frac{1}{R} \frac{\partial I_{\omega_n}(\eta_m R)}{\partial r} - \frac{\omega_n^2}{R^2} I_{\omega_n}(\eta_m R) - \frac{\partial^2 I_{\omega_n}(\eta_m R)}{\partial r^2} \right] \right. \right. \right. \\ & \left. \left. \left. - \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)} \Gamma_{sm}^{(i)}}{2 \gamma_s^2} \left[\frac{\omega_n^2}{R^2} J_{\omega_n}(\gamma_s R) + \frac{\partial^2 J_{\omega_n}(\gamma_s R)}{\partial r^2} \right] \right\} \frac{\cos(\eta_m z)}{\sin(\eta_m z)} \frac{\sin(\omega_n \theta)}{\cos(\omega_n \theta)} \right\} \end{aligned} \tag{37}$$

where

$$\mathfrak{R}_{0n}^{(l)}(R) = \sum_{s=1}^{\infty} \left\{ \left[\left(\frac{\omega_n^2}{\gamma_s^2 R^2} - 1 \right) C_{sn}^{(l)} - 2\nu D_{sn}^{(l)} \right] \frac{\sinh(\gamma_s L)}{\gamma_s L} + D_{sn}^{(l)} \cosh(\gamma_s L) \left(\frac{\omega_n^2}{\gamma_s^2 R^2} - 1 \right) \right\} J_{\omega_n}(\gamma_s R) \tag{38}$$

$$\mathfrak{Z}_{0n}^{(l)}(R) = \sum_{s=1}^{\infty} \frac{F_{sn}^{(l)} \omega_n}{\gamma_s^2 R^2} \frac{\sinh(\gamma_s L)}{\gamma_s L} J_{\omega_n}(\gamma_s R) \tag{39}$$

$$\mathfrak{N}_{0n}^{(l)}(R) = \sum_{s=1}^{\infty} \frac{F_{sn}^{(l)} \omega_n}{2 \gamma_s R} \frac{\sinh(\gamma_s L)}{\gamma_s L} J_{\omega_n}(\gamma_s R) \tag{40}$$

$$\mathfrak{P}_{0n}^{(l)}(R) = \sum_{s=1}^{\infty} \frac{\omega_n^2}{\gamma_s^2 R^2} \left[C_{sn}^{(l)} \frac{\sinh(\gamma_s L)}{\gamma_s L} + D_{sn}^{(l)} \cosh(\gamma_s L) \right] J_{\omega_n}(\gamma_s R) \tag{41}$$

$$\hat{\lambda}_{0n}^{(l)}(R) = \sum_{s=1}^{\infty} \frac{F_{sm}^{(l)}}{2\gamma_s^2} \left[\frac{\omega_n^2}{R^2} J_{\omega_n}(\gamma_s R) + \frac{\partial^2 J_{\omega_n}(\gamma_s R)}{\partial r^2} \right] \frac{\sinh(\gamma_s L)}{\gamma_s L} \tag{42}$$

where $l = 1, 2$. In Eqs. (35)–(37), $\Gamma_{sm}^{(i)}$ ($i = 1, 2$) are the coefficients for the Fourier expansion of $\cosh(\gamma_s z)$ and $\sinh(\gamma_s z)$, respectively:

$$\Gamma_{sm}^{(1)} = \frac{2\gamma_s (-1)^m \sinh(\gamma_s L)}{L(\gamma_s^2 + \eta_m^2)} \tag{43}$$

$$\Gamma_{sm}^{(2)} = \frac{2\eta_m (-1)^{m+1} \sinh(\gamma_s L)}{L(\gamma_s^2 + \eta_m^2)} \tag{44}$$

While $A_{sm}^{(i)}$ ($i = 1, 2$) are the coefficients for the Fourier expansion of $\gamma_s z \sinh(\gamma_s z)$ and $\gamma_s z \cosh(\gamma_s z)$, respectively:

$$A_{sm}^{(1)} = \frac{2\gamma_s (-1)^m \sinh(\gamma_s L)}{L(\gamma_s^2 + \eta_m^2)} \left[\gamma_s L \cosh(\gamma_s L) + \frac{\eta_m^2 - \gamma_s^2}{\gamma_s^2 + \eta_m^2} \sinh(\gamma_s L) \right] \tag{45}$$

$$A_{sm}^{(2)} = \frac{2\eta_m \gamma_s (-1)^{m+1} \sinh(\gamma_s L)}{\gamma_s^2 + \eta_m^2} \tag{46}$$

Equating coefficients of the Fourier expansions in z and θ in Eqs. (35)–(37) to those corresponding to BC11, BC12 and BC13 given by Eq. (31), we obtain a system of equations relating the unknown constants.

In particular, the following equations are obtained by BC11:

$$(2\nu - 1)C_{00} + \nu A_{00} = a_{00}^{(r)}/4 \tag{47}$$

$$-\omega_n(\omega_n - 1) \left(A_{0n}^{(1)} + E_{0n}^{(1)} \right) R^{\omega_n - 2} - \mathfrak{R}_{0n}^{(1)}(R) + \mathfrak{I}_{0n}^{(1)}(R) = a_{0n}^{(r)}/2 \tag{48}$$

$$-\omega_n(\omega_n - 1) \left(A_{0n}^{(2)} - E_{0n}^{(2)} \right) R^{\omega_n - 2} - \mathfrak{R}_{0n}^{(2)}(R) - \mathfrak{I}_{0n}^{(2)}(R) = c_{0n}^{(r)}/2 \tag{49}$$

$$\begin{aligned}
 & \alpha_{1k} \left\{ A_{mn}^{(k)} \left[2\nu I_{\omega_n}(\eta_m R) - \frac{2}{\eta_m^2} \frac{\partial^2 I_{\omega_n}(\eta_m R)}{\partial r^2} - \frac{R}{\eta_m^2} \frac{\partial^3 I_{\omega_n}(\eta_m R)}{\partial r^3} \right] - \frac{B_{mn}^{(k)}}{\eta_m^2} \frac{\partial^2 I_{\omega_n}(\eta_m R)}{\partial r^2} \right\} \\
 & + \alpha_{2k} \left\{ \frac{E_{mn}^{(k)} \omega_n}{\eta_m^2} \left[\frac{1}{R} \frac{\partial I_{\omega_n}(\eta_m R)}{\partial r} - \frac{I_{\omega_n}(\eta_m R)}{R^2} \right] \right\} + \sum_{s=1}^{\infty} \left\{ \left\{ \frac{\omega_n^2}{\gamma_s^2 R^2} \left[(C_{sn}^{(k)} + D_{sn}^{(k)}) \Gamma_{sm}^{(i)} + D_{sn}^{(k)} A_{sm}^{(i)} \right] \right. \right. \\
 & \left. \left. - \left[(C_{sn}^{(k)} + (2\nu + 1) D_{sn}^{(k)}) \Gamma_{sm}^{(i)} + D_{sn}^{(k)} A_{sm}^{(i)} \right] \right\} J_{\omega_n}(\gamma_s R) - \alpha_{2k} \frac{F_{sn}^{(k)} \omega_n}{\gamma_s^2} \Gamma_{sm}^{(i)} \frac{J_{\omega_n}(\gamma_s R)}{R^2} \right\} \\
 & = \lambda_{mn} \Omega_{mn}^{(r)} \tag{50}
 \end{aligned}$$

where α_{ik} ($i = 1, 2; k = 1, 2, 3, 4$) and $\Omega_{mn}^{(r)}$ are defined in Table 4.

The following equations are obtained by BC12:

$$9(1 - \nu)E_{00} - \mathfrak{N}_{0n}^{(1)}(R) = a_{0n}^{(z)}/2 \tag{51}$$

$$\begin{aligned}
 & \frac{A_{mn}^{(k)}}{\eta_m} \left[(3 - 2\nu) \frac{\partial I_{\omega_n}(\eta_m R)}{\partial r} + R \frac{\partial^2 I_{\omega_n}(\eta_m R)}{\partial r^2} \right] + \frac{B_{mn}^{(k)}}{\eta_m} \frac{\partial I_{\omega_n}(\eta_m R)}{\partial r} + \beta_{1k} \frac{E_{mn}^{(k)} \omega_n}{2\eta_m R} I_{\omega_n}(\eta_m R) \\
 & + \beta_{2k} \sum_{s=1}^{\infty} \frac{F_{sn}^{(k)} \omega_n}{2\gamma_s R} J_{\omega_n}(\gamma_s R) \Gamma_{sm}^{(i)} \\
 & = \lambda_{mn} \Omega_{mn}^{(z)} \tag{52}
 \end{aligned}$$

where β_{ik} ($i = 1, 2; k = 1, 2, 3, 4$) and $\Omega_{mn}^{(z)}$ are defined in Table 5.

The following equations are obtained by BC13:

$$\omega_n(\omega_n - 1) \left(A_{0n}^{(1)} - E_{0n}^{(1)} \right) R^{\omega_n - 2} - \wp_{0n}^{(1)}(R) + \lambda_{0n}^{(1)}(R) = c_{0n}^{(\theta)}/2 \tag{53}$$

$$-\omega_n(\omega_n - 1) \left(A_{0n}^{(2)} + E_{0n}^{(2)} \right) R^{\omega_n - 2} + \wp_{0n}^{(2)}(R) + \lambda_{0n}^{(2)}(R) = a_{0n}^{(\theta)}/2 \tag{54}$$

Table 4

The definitions of α_{ik} ($i = 1, 2$), and $\Omega_{mn}^{(r)}$ used in Eq. (50) for $k = 1, 2, 3$, and 4

| k | α_{1k} | α_{2k} | $\Omega_{mn}^{(r)}$ |
|-----|---------------|---------------|---------------------|
| 1 | +1 | -1 | $d_{mn}^{(r)}$ |
| 2 | +1 | +1 | $c_{mn}^{(r)}$ |
| 3 | -1 | -1 | $b_{mn}^{(r)}$ |
| 4 | -1 | +1 | $d_{mn}^{(r)}$ |

Table 5
The definitions of β_{ik} ($i = 1, 2$) and $\Omega_{mn}^{(z)}$ used in Eq. (52) for $k = 1, 2, 3$, and 4

| k | β_{1k} | β_{2k} | $\Omega_{mn}^{(z)}$ |
|-----|--------------|--------------|---------------------|
| 1 | +1 | -1 | $b_{mn}^{(z)}$ |
| 2 | -1 | +1 | $d_{mn}^{(z)}$ |
| 3 | -1 | -1 | $a_{mn}^{(z)}$ |
| 4 | +1 | +1 | $c_{mn}^{(z)}$ |

$$\begin{aligned} &\kappa_{1k} \left\{ \frac{\omega_n A_{mn}^{(k)}}{\eta_m^2} \frac{\partial^2 I_{\omega_n}(\eta_m R)}{\partial r^2} + \frac{\omega_n B_{mn}^{(k)}}{\eta_m^2} \left[\frac{1}{R} \frac{\partial I_{\omega_n}(\eta_m R)}{\partial r} - \frac{1}{R^2} I_{\omega_n}(\eta_m R) \right] \right\} + \frac{E_{mn}^{(k)}}{2\eta_m^2} \left[\frac{1}{R} \frac{\partial I_{\omega_n}(\eta_m R)}{\partial r} \right. \\ &\quad \left. - \frac{\omega_n^2}{R^2} I_{\omega_n}(\eta_m R) - \frac{\partial^2 I_{\omega_n}(\eta_m R)}{\partial r^2} \right] + \sum_{s=1}^{\infty} \left\{ \kappa_{2k} \frac{\omega_n}{\gamma_s^2 R^2} \left[(C_{sn}^{(k)} + D_{sn}^{(k)}) \Gamma_{sm}^{(i)} + D_{sn}^{(k)} A_{sm}^{(i)} \right] J_{\omega_n}(\gamma_s R) \right. \\ &\quad \left. - \frac{F_{sn}^{(k)} \Gamma_{sm}^{(i)}}{2\gamma_s^2} \left[\frac{\omega_n^2}{R^2} J_{\omega_n}(\gamma_s R) + \frac{\partial^2 J_{\omega_n}(\gamma_s R)}{\partial r^2} \right] \right\} \\ &= \lambda_{mn} \Omega_{mn}^{(\theta)} \end{aligned} \tag{55}$$

where κ_{ik} ($i = 1, 2; k = 1, 2, 3, 4$) and $\Omega_{mn}^{(\theta)}$ are defined in Table 6.

On the end surface, $z = L$, Eq. (9) can be rewritten as

$$\sigma_{zr} = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \zeta_{sn}^+ [e_{sn}^+ \sin(\omega_n \theta) + f_{sn}^+ \cos(\omega_n \theta)] \frac{J_{\omega_n}(\gamma_s r)}{r} \tag{56}$$

$$\sigma_{zz} = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \zeta_{sn}^+ [g_{sn}^+ \sin(\omega_n \theta) + h_{sn}^+ \cos(\omega_n \theta)] J_{\omega_n}(\gamma_s r) \tag{57}$$

$$\sigma_{z\theta} = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \zeta_{sn}^+ [k_{sn}^+ \sin(\omega_n \theta) + l_{sn}^+ \cos(\omega_n \theta)] \frac{J_{\omega_n}(\gamma_s r)}{r} \tag{58}$$

On the other end surface $z = -L$, Eq. (10) is rewritten as

Table 6
The definitions of κ_{ik} ($i = 1, 2$) and $\Omega_{mn}^{(\theta)}$ used in Eq. (55) for $k = 1, 2, 3$, and 4

| k | κ_{1k} | κ_{2k} | $\Omega_{mn}^{(\theta)}$ |
|-----|---------------|---------------|--------------------------|
| 1 | +1 | +1 | $c_{mn}^{(\theta)}$ |
| 2 | -1 | -1 | $a_{mn}^{(\theta)}$ |
| 3 | -1 | +1 | $d_{mn}^{(\theta)}$ |
| 4 | +1 | -1 | $b_{mn}^{(\theta)}$ |

$$\sigma_{rz} = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \zeta_{sn} [e_{sn}^- \sin(\omega_n \theta) + f_{sn}^- \cos(\omega_n \theta)] \frac{J_{\omega_n}(\gamma_s r)}{r} \quad (59)$$

$$\sigma_{zz} = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \zeta_{sn} [g_{sn}^- \sin(\omega_n \theta) + h_{sn}^- \cos(\omega_n \theta)] J_{\omega_n}(\gamma_s r) \quad (60)$$

$$\sigma_{z\theta} = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \zeta_{sn} [k_{sn}^- \sin(\omega_n \theta) + l_{sn}^- \cos(\omega_n \theta)] \frac{J_{\omega_n}(\gamma_s r)}{r} \quad (61)$$

where

$$\zeta_{sn} = \begin{cases} 1/2 & \text{for } n = 0 \\ 1 & \text{for } n \neq 0 \end{cases} \quad (62)$$

$$e_{sn}^+ = \frac{4\lambda_s^2}{R^2 T (\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \int_0^R \int_{-T/2}^{T/2} r^2 f_{2r}(r, \theta) J_{\omega_n}(\gamma_s r) \sin(\omega_n \theta) \, d\theta \, dr \quad (63)$$

$$f_{sn}^+ = \frac{4\lambda_s^2}{R^2 T (\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \int_0^R \int_{-T/2}^{T/2} r^2 f_{2r}(r, \theta) J_{\omega_n}(\gamma_s r) \cos(\omega_n \theta) \, d\theta \, dr \quad (64)$$

$$g_{sn}^+ = \frac{4\lambda_s^2}{R^2 T (\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \int_0^R \int_{-T/2}^{T/2} r f_{2z}(r, \theta) J_{\omega_n}(\gamma_s r) \sin(\omega_n \theta) \, d\theta \, dr \quad (65)$$

$$h_{sn}^+ = \frac{4\lambda_s^2}{R^2 T (\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \int_0^R \int_{-T/2}^{T/2} r f_{2z}(r, \theta) J_{\omega_n}(\gamma_s r) \cos(\omega_n \theta) \, d\theta \, dr \quad (66)$$

$$k_{zn}^+ = \frac{4\lambda_s^2}{R^2 T (\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \int_0^R \int_{-T/2}^{T/2} r^2 f_{2\theta}(r, \theta) J_{\omega_n}(\gamma_s r) \sin(\omega_n \theta) \, d\theta \, dr \quad (67)$$

$$l_{sn}^+ = \frac{4\lambda_s^2}{R^2 T (\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \int_0^R \int_{-T/2}^{T/2} r^2 f_{2\theta}(r, \theta) J_{\omega_n}(\gamma_s r) \cos(\omega_n \theta) \, d\theta \, dr \quad (68)$$

e_{sn}^- , f_{sn}^- , g_{sn}^- , h_{sn}^- , k_{sn}^- and l_{sn}^- can be obtained by replacing superscript '+' by '-' and ' $f_{2\alpha}$ ' by ' $f_{3\alpha}$ ' ($\alpha = r, z, \theta$) in Eqs. (63)–(68).

It can be shown that once one end boundary (say $z = L$) is satisfied, the other end boundary will be satisfied automatically. To see this, let us consider σ_{zz} as an example and rewrite Eqs. (57) and (60) as

$$\sigma_{zz} = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2} \zeta_{sn} \{ [(g_{sn}^+ + g_{sn}^-) \sin(\omega_n \theta) + (h_{sn}^+ + h_{sn}^-) \cos(\omega_n \theta)] + [(g_{sn}^+ - g_{sn}^-) \sin(\omega_n \theta) + (h_{sn}^+ - h_{sn}^-) \cos(\omega_n \theta)] \} J_{\omega_n}(\gamma_s r) \tag{69}$$

$$\sigma_{zz} = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2} \zeta_{sn} \{ [(g_{sn}^+ + g_{sn}^-) \sin(\omega_n \theta) + (h_{sn}^+ + h_{sn}^-) \cos(\omega_n \theta)] - [(g_{sn}^+ - g_{sn}^-) \sin(\omega_n \theta) + (h_{sn}^+ - h_{sn}^-) \cos(\omega_n \theta)] \} J_{\omega_n}(\gamma_s r) \tag{70}$$

on $z = L$ and $z = -L$, respectively.

Comparing Eqs. (69) and (70), we found that the term inside the first bracket [] in Eqs. (69) and (70) is an even function with respect to z , while the terms inside the second bracket [] is an odd function with respect to z . Similar consideration can also be made to $\sigma_{z\theta}$ and σ_{zr} .

To consider BC21, BC22, and BC23, the internal stress field given in Eqs. (27)–(29) is first expanded into Fourier–Bessel series

$$\begin{aligned} \sigma_{zz} = & 2(2 - \nu)C_{00} + (1 - \nu)A_{00} + \sum_{n=0}^{\infty} \left\{ \sum_{s=1}^{\infty} \sum_{m=1}^{\infty} \left\{ A_{mn}^{(k)} [(4 - 2\nu - \omega_n)T_{sn} + U_{sn}] \right. \right. \\ & \left. \left. + B_{mn}^{(k)} T_{sn} \right\} \begin{matrix} \cos(\eta_m z) \\ -\sin(\eta_m z) \end{matrix} - \left[(C_{sn}^{(k)} + (2\nu - 1)D_{sn}^{(k)}) \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} + D_{sn}^{(k)} \gamma_s z \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} \right] \right. \\ & \left. \times \begin{matrix} J_{\omega_n}(\gamma_s r) \cos(\omega_n \theta) \\ \sin(\omega_n \theta) \end{matrix} \right\} \end{aligned} \tag{71}$$

$$\begin{aligned} \sigma_{zr} = & \sum_{s=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \sum_{m=1}^{\infty} \left\{ \left[\frac{A_{mn}^{(k)}}{\eta_m} [(\omega_n^2 - 2\omega_n(1 - \nu))T_{sm} + U_{sm} + W_{sm}] + \frac{B_{mn}^{(k)}}{\eta_m} [U_{sm} - \omega_n T_{sm}] \right. \right. \right. \\ & \left. \left. \times \begin{matrix} \sin(\eta_m z) \cos(\omega_n \theta) - \frac{E_{mn}^{(k)} \omega_n}{2\eta_m} T_{sm} - \sin(\eta_m z) \cos(\omega_n \theta) \\ \cos(\eta_m z) \sin(\omega_n \theta) - \sin(\omega_n \theta) \end{matrix} \right\} \right. \\ & \left. + \frac{F_{sn}^{(k)} \omega_n}{2\gamma_s} \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} \frac{\cos(\omega_n \theta)}{-\sin(\omega_n \theta)} - \sum_{p=1}^{\infty} \frac{V_{sn}}{\gamma_p} \left[(C_{pn}^{(k)} + 2\nu D_{pn}^{(k)}) \frac{\sinh(\gamma_p z)}{\cosh(\gamma_p z)} \right. \right. \\ & \left. \left. + D_{pn}^{(k)} \gamma_p z \frac{\cosh(\gamma_p z)}{\sinh(\gamma_p z)} \right] \frac{\cos(\omega_n \theta)}{\sin(\omega_n \theta)} \right\} \frac{J_{\omega_n}(\gamma_s r)}{r} \end{aligned} \tag{72}$$

$$\begin{aligned}
\sigma_{z\theta} = & \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\left\{ \frac{\omega_n A_{mn}^{(k)}}{\eta_m} [(2 - 2\nu - \omega_n) T_{sm} + U_{sm}] + \frac{\omega_n B_{mn}^{(k)}}{\eta_m} T_{sm} \right\} \frac{\sin(\eta_m z) - \sin(\omega_n \theta)}{\cos(\eta_m z) \cos(\omega_n \theta)} \right. \right. \\
& + \left. \left. \frac{E_{mn}^{(k)}}{2\eta_m} [U_{sm} - \omega_n T_{sm}] \frac{-\sin(\eta_m z)}{\cos(\eta_m z)} \frac{\sin(\omega_n \theta)}{\cos(\omega_n \theta)} \right\} - \frac{\omega_n}{\gamma_s} \left[(C_{sn}^{(k)} + 2\nu D_{sn}^{(k)}) \frac{\sinh(\gamma_s z)}{\cosh(\gamma_s z)} \right. \right. \\
& + \left. \left. D_{sn}^{(k)} \gamma_s z \frac{\cosh(\gamma_s z)}{\sinh(\gamma_s z)} \right] \frac{-\sin(\omega_n \theta)}{\cos(\omega_n \theta)} + \sum_{p=1}^{\infty} \frac{F_{pn}^{(k)} V_{sn}}{2\gamma_p} \frac{\sinh(\gamma_p z)}{\cosh(\gamma_p z)} \frac{\sin(\omega_n \theta)}{\cos(\omega_n \theta)} \right\} \frac{J_{\omega_n}(\gamma_s r)}{r} \quad (73)
\end{aligned}$$

where

$$T_{sm} = \frac{2\lambda_s^2}{(\lambda_s^2 - \omega_n^2) [(\eta_m R)^2 + \lambda_s^2]} J_{\omega_n}^2(\lambda_s) [\eta_m R I_{\omega_n+1}(\eta_m R) J_{\omega_n}(\lambda_s) + \lambda_s I_{\omega_n}(\eta_m R) J_{\omega_n+1}(\lambda_s)] \quad (74)$$

$$\begin{aligned}
U_{sm} = & \frac{2\lambda_s^2 \eta_m R}{(\lambda_s^2 - \omega_n^2) [(\eta_m R)^2 + \lambda_s^2]} J_{\omega_n}^2(\lambda_s) \left\{ \frac{2(\omega_n - 1) \eta_m R}{(\eta_m R)^2 + \lambda_s^2} [\eta_m R I_{\omega_n+1}(\eta_m R) J_{\omega_n}(\lambda_s) \right. \\
& + \left. \lambda_s I_{\omega_n}(\eta_m R) J_{\omega_n+1}(\lambda_s)] + \frac{2\omega_n \lambda_s}{(\eta_m R)^2 + \lambda_s^2} [\eta_m R I_{\omega_n}(\eta_m R) J_{\omega_n-1}(\lambda_s) + \lambda_s I_{\omega_n-1}(\eta_m R) J_{\omega_n}(\lambda_s)] \right. \\
& \left. + \eta_m R I_{\omega_n}(\eta_m R) J_{\omega_n}(\lambda_s) - \lambda_s I_{\omega_n-1}(\eta_m R) J_{\omega_n-1}(\lambda_s) \right\} \quad (75)
\end{aligned}$$

$$\begin{aligned}
W_{sm} = & \frac{2\lambda_s^2 \eta_m^2 R^2}{(\lambda_s^2 - \omega_n^2) [(\eta_m R)^2 + \lambda_s^2]} J_{\omega_n}^2(\lambda_s) \left\{ \eta_m R I_{\omega_n+1}(\eta_m R) J_{\omega_n}(\lambda_s) + \lambda_s I_{\omega_n}(\eta_m R) J_{\omega_n+1}(\lambda_s) \right. \\
& + \frac{2}{(\eta_m R)^2 + \lambda_s^2} \left\{ \frac{2(\nu + 1)(\lambda_s^2 - \eta_m^2 R^2)}{(\eta_m R)^2 + \lambda_s^2} [\eta_m R I_{\omega_n+1}(\eta_m R) J_{\omega_n}(\lambda_s) + \lambda_s I_{\omega_n}(\eta_m R) J_{\omega_n+1}(\lambda_s)] \right. \\
& + \frac{4\nu \eta_m R \lambda_s}{(\eta_m R)^2 + \lambda_s^2} [\eta_m R I_{\omega_n+2}(\eta_m R) J_{\omega_n+1}(\lambda_s) + \lambda_s I_{\omega_n+1}(\eta_m R) J_{\omega_n+2}(\lambda_s)] \\
& \left. \left. + \lambda_s \eta_m R I_{\omega_n+1}(\eta_m R) J_{\omega_n+1}(\lambda_s) - (\lambda_s^2 - \eta_m^2 R^2) I_{\omega_n}(\eta_m R) J_{\omega_n}(\lambda_s) \right\} \right\} \quad (76)
\end{aligned}$$

If $\lambda_p \neq \lambda_s$, then

$$\begin{aligned}
 V_{sn} = & \frac{2\lambda_s^2}{(\lambda_s^2 - \omega_n^2)(\lambda_p^2 - \lambda_s^2)J_{\omega_n}^2(\lambda_s)} \left\{ \lambda_p^2 J_{\omega_n}(\lambda_p) J_{\omega_n}(\lambda_s) + \lambda_s \lambda_p J_{\omega_n-1}(\lambda_p) J_{\omega_n-1}(\lambda_s) \right. \\
 & - \frac{2\lambda_p \lambda_s \omega_n}{\lambda_p^2 - \lambda_s^2} [\lambda_p J_{\omega_n}(\lambda_p) J_{\omega_n-1}(\lambda_s) - \lambda_s J_{\omega_n-1}(\lambda_p) J_{\omega_n}(\lambda_s)] \\
 & \left. + \frac{\omega_n(\lambda_p^2 + \lambda_s^2) - 2\lambda_p^2}{\lambda_p^2 - \lambda_s^2} [\lambda_p J_{\omega_n+1}(\lambda_p) J_{\omega_n}(\lambda_s) - \lambda_s J_{\omega_n}(\lambda_p) J_{\omega_n+1}(\lambda_s)] \right\} \quad (77)
 \end{aligned}$$

If $\lambda_p = \lambda_s$, then

$$V_{sn} = \frac{\lambda_s^2 J_{\omega_n-1}(\lambda_s) J_{\omega_n+1}(\lambda_s)}{(\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \quad (78)$$

The proofs of these formulas are given in Appendix A. By substituting $z = L$ into Eq. (71) and comparing with the corresponding coefficients of Eq. (69), we have

$$2(2 - \nu)C_{00} + (1 - \nu)A_{00} = 0 \quad (79)$$

$$\begin{aligned}
 & \sum_{m=1}^{\infty} \left\{ A_{mn}^{(1)} [(4 - 2\nu - \omega_n) T_{sm} + U_{sm}] + B_{mn}^{(1)} T_{sm} \right\} (-1)^m \\
 & - \left[(C_{sn}^{(1)} + (2\nu - 1)D_{sn}^{(1)}) \cosh(\gamma_s L) + D_{sn}^{(1)} \gamma_s L \sinh(\gamma_s L) \right] = \frac{1}{2} \zeta_{sn} (h_{sn}^+ + h_{sn}^-) \quad (80)
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{m=1}^{\infty} \left\{ A_{mn}^{(2)} [(4 - 2\nu - \omega_n) T_{sm} + U_{sm}] + B_{mn}^{(2)} T_{sm} \right\} (-1)^m - \left[(C_{sn}^{(2)} + (2\nu - 1)D_{sn}^{(2)}) \cosh(\gamma_s L) \right. \\
 & \left. + D_{sn}^{(2)} \gamma_s L \sinh(\gamma_s L) \right] \\
 & = \frac{1}{2} \zeta_{sn} (g_{sn}^+ + g_{sn}^-) \quad (81)
 \end{aligned}$$

$$(C_{sn}^{(3)} + (2\nu - 1)D_{sn}^{(3)}) \sinh(\gamma_s L) + D_{sn}^{(3)} \gamma_s L \cosh(\gamma_s L) = \frac{1}{2} \zeta_{sn} (h_{sn}^+ - h_{sn}^-) \quad (82)$$

$$(C_{sn}^{(4)} + (2\nu - 1)D_{sn}^{(4)}) \sinh(\gamma_s L) + D_{sn}^{(4)} \gamma_s L \cosh(\gamma_s L) = \frac{1}{2} \zeta_{sn} (g_{sn}^+ - g_{sn}^-) \quad (83)$$

The same system of equation is obtained if we substitute $z = -L$ into Eq. (71) and comparing the corresponding coefficients of Eq. (70). Thus, only one of the end boundaries (either BC22 or BC32) has to be satisfied by the stress σ_{zz} given in Eq. (71). Similar consideration can also be made for $\sigma_{z\theta}$ and σ_{zr} and leads to the same conclusion.

Applying BC21 or BC31 into Eq. (72), we have

$$\frac{F_{sn}^{(1)}\omega_n}{2\gamma_s}\sinh(\gamma_p L) - \sum_{p=1}^{\infty} \frac{V_{pn}}{\gamma_p} \left[\left(C_{pn}^{(1)} + 2\nu D_{pn}^{(1)} \right) \sinh(\gamma_p L) + D_{pn}^{(1)} \gamma_p L \cosh(\gamma_p L) \right] = \frac{1}{2} \zeta_{sn} (f_{sn}^+ + f_{sn}^-) \quad (84)$$

$$- \frac{F_{sn}^{(2)}\omega_n}{2\gamma_s}\sinh(\gamma_p L) - \sum_{p=1}^{\infty} \frac{V_{pn}}{\gamma_p} \left[\left(C_{pn}^{(2)} + 2\nu D_{pn}^{(2)} \right) \sinh(\gamma_p L) + D_{pn}^{(2)} \gamma_p L \cosh(\gamma_p L) \right] = \frac{1}{2} \zeta_{sn} (e_{sn}^+ + e_{sn}^-) \quad (85)$$

$$\begin{aligned} & \sum_{m=1}^{\infty} \left\{ \left[\frac{A_{mn}^{(3)}}{\eta_m} \left[(\omega_n^2 - 2\omega_n(1-\nu))T_{sm} + U_{sm} + W_{sm} \right] + \frac{B_{mn}^{(3)}}{\eta_m} [U_{sm} - \omega_n T_{sm}] \right] - \frac{E_{mn}^{(3)}\omega_n}{2\eta_m} T_{sm} \right\} \\ & \times (-1)^m + \frac{F_{sn}^{(3)}\omega_n}{2\gamma_s} \cosh(\gamma_s L) - \sum_{p=1}^{\infty} \frac{V_{sn}}{\gamma_p} \left[\left(C_{pn}^{(3)} + 2\nu D_{pn}^{(3)} \right) \cosh(\gamma_p L) + D_{pn}^{(3)} \gamma_p L \sinh(\gamma_p L) \right] \\ & = \frac{1}{2} \zeta (f_{sn}^+ - f_{sn}^-) \end{aligned} \quad (86)$$

$$\begin{aligned} & \sum_{m=1}^{\infty} \left\{ \left[\frac{A_{mn}^{(4)}}{\eta_m} \left[(\omega_n^2 - 2\omega_n(1-\nu))T_{sm} + U_{sm} + W_{sm} \right] + \frac{B_{mn}^{(4)}}{\eta_m} [U_{sm} - \omega_n T_{sm}] \right] + \frac{E_{mn}^{(4)}\omega_n}{2\eta_m} T_{sm} \right\} \\ & \times (-1)^m - \frac{F_{sn}^{(4)}\omega_n}{2\gamma_s} \cosh(\gamma_s L) - \sum_{p=1}^{\infty} \frac{V_{sn}}{\gamma_p} \left[\left(C_{pn}^{(4)} + 2\nu D_{pn}^{(4)} \right) \cosh(\gamma_p L) + D_{pn}^{(4)} \gamma_p L \sinh(\gamma_p L) \right] \\ & = \frac{1}{2} \zeta (e_{sn}^+ - e_{sn}^-) \end{aligned} \quad (87)$$

Finally, applying BC23 or BC33 into Eq. (73), we have

$$\frac{\omega_n}{\gamma_s} \left[\left(C_{sn}^{(1)} + 2\nu D_{sn}^{(1)} \right) \sinh(\gamma_s L) + D_{sn}^{(1)} \gamma_s L \cosh(\gamma_s L) \right] + \sum_{p=1}^{\infty} \frac{F_{pn}^{(1)} V_{sn}}{2\gamma_p} \sinh(\gamma_p L) = \frac{1}{2} \zeta_{sn} (k_{sn}^+ + k_{sn}^-) \quad (88)$$

$$- \frac{\omega_n}{\gamma_s} \left[\left(C_{sn}^{(2)} + 2\nu D_{sn}^{(2)} \right) \sinh(\gamma_s L) + D_{sn}^{(2)} \gamma_s L \cosh(\gamma_s L) \right] + \sum_{p=1}^{\infty} \frac{F_{pn}^{(2)} V_{sn}}{2\gamma_p} \sinh(\gamma_p L) = \frac{1}{2} \zeta_{sn} (l_{sn}^+ + l_{sn}^-) \quad (89)$$

$$\begin{aligned} & \sum_{m=1}^{\infty} \left\{ - \left[\frac{\omega_n A_{mn}^{(3)}}{\eta_m} \left[(2 - 2\nu - \omega_n)T_{sm} + U_{sm} \right] + \frac{\omega_n B_{mn}^{(3)}}{\eta_m} T_{sm} \right] + \frac{E_{mn}^{(3)}}{2\eta_m} [U_{sm} - \omega_n T_{sm}] \right\} (-1)^m \\ & + \frac{\omega_n}{\gamma_s} \left[\left(C_{sn}^{(3)} + 2\nu D_{sn}^{(3)} \right) \cosh(\gamma_s L) + D_{sn}^{(3)} \gamma_s L \sinh(\gamma_s L) \right] + \sum_{p=1}^{\infty} \frac{F_{pn}^{(3)} V_{sn}}{2\gamma_p} \cosh(\gamma_p L) \\ & = \frac{1}{2} \zeta (k_{sn}^+ - k_{sn}^-) \end{aligned} \quad (90)$$

$$\begin{aligned}
& \sum_{m=1}^{\infty} \left\{ \left[\frac{\omega_n A_{mn}^{(4)}}{\eta_m} (2 - 2\nu - \omega_n) T_{sm} + U_{sm} \right] + \frac{\omega_n B_{mn}^{(4)}}{\eta_m} T_{sm} \right\} + \frac{E_{mn}^{(4)}}{2\eta_m} [U_{sm} - \omega_n T_{sm}] \Big\} (-1)^m \\
& - \frac{\omega_n}{\gamma_s} \left[\left(C_{sn}^{(4)} + 2\nu D_{sn}^{(4)} \right) \cosh(\gamma_s L) + D_{sn}^{(4)} \gamma_s L \sinh(\gamma_s L) \right] + \sum_{p=1}^{\infty} \frac{F_{pn}^{(4)} V_{sn}}{2\gamma_p} \cosh(\gamma_s z) \\
& = \frac{1}{2} \zeta (l_{sn}^+ - l_{sn}^-)
\end{aligned} \tag{91}$$

Therefore, A_{00} and C_{00} can be obtained from Eqs. (47) and (79); E_{00} can be obtained from Eq. (51); $A_{0n}^{(l)}$ and $E_{0n}^{(l)}$ ($l = 1, 2$) can be obtained from Eqs. (48), (49), (53) and (54); for each n , $12(M + S)$ unknown constants $A_{mn}^{(k)}$, $B_{mn}^{(k)}$, $E_{mn}^{(k)}$, $C_{sn}^{(k)}$, $D_{sn}^{(k)}$, and $F_{sn}^{(k)}$ ($k = 1, 2, 3, 4$; $m = 1, 2, \dots, M$; $s = 1, 2, \dots, S$) can be solved from the system of equations forming from Eqs. (50), (52) and (55) with m from 1 to M , and Eqs. (80)–(91) with s from 1 to S . Thus, we have exactly $12(M + S)$ equations for $12(M + S)$ unknowns $A_{mn}^{(k)}$, $B_{mn}^{(k)}$, $E_{mn}^{(k)}$, $C_{sn}^{(k)}$, $D_{sn}^{(k)}$, and $F_{sn}^{(k)}$ for each n . If the summation n is from 1 to N , the total number of unknowns and equations are $3 + 4N + 12N(M + S)$. Finally, all stress components can be obtained by back substituting all these coefficients into Eqs. (25)–(30). Therefore, the general elastic solution for stresses within a finite isotropic solid cylinder subjected to arbitrary surface loads is obtained, and some special cases of this general solution will be discussed next.

7. Special cases

Let us consider some special loading cases of our general solution given in the previous section. When we set $\omega_n = 2n$ ($n = 1, 2, \dots$) and $k = l = 1$, our general solution reduces to the solution for finite cylinders under the diametral Point Load Strength Test (PLST) considered by Chau and Wei (1999); when we set $\omega_n = 2n$ ($n = 1, 2, \dots$), $k = l = 1$, and $C_{sn}^{(1)} = D_{sn}^{(1)} = F_{sn}^{(1)} = 0$, our general solution reduces to the solution for finite cylinders with zero shear displacements on the end surfaces under the diametral PLST considered by Chau (1998a, 1998b); when we set $\omega_n = 0$, $k = l = 1$, and $E_{mn}^{(1)} = F_{sn}^{(1)} = 0$, our general solution reduces to the solution for finite cylinders under the axial PLST considered by Wei et al. (1999); and, when we set $\omega_n = 0$ and $E_{mn}^{(1)} = F_{sn}^{(1)} = 0$, our general solution reduces to the general solution for finite cylinders under axisymmetric load by Saito (1954, 1952). In addition, the solutions for finite cylinders with fully or partially constrained radial displacement at the end surfaces under confined or unconfined compression tests considered by Filon (1902) and Watanabe (1996) can also be obtained by simply replacing the corresponding traction boundary with the displacement boundary.

8. Characteristics of the roots of $J'_{\omega_n}(\lambda_s) = 0$

When $\omega_n = 0$ (or cylinders under axisymmetric loads), Saito (1952) has considered the roots of $J'_0(\lambda_s) = 0$ in obtaining his numerical solution. These roots have also been used by various authors, including Ogaki and Nakajima (1983), Watanabe (1996) and Wei et al. (1999).

For obtaining the analytic solution for the diametral PLST test, Chau and Wei (1999) has evaluated the roots of $J'_{2n}(\lambda_s) = 0$ ($n = 0, 1, 2, \dots$). Since the accuracy of the present approach also depends on the accuracy of the roots λ_s , in this section we will consider some general characteristics of the roots of $J'_{\omega_n}(\lambda_s) = 0$, so that these roots can be found accurately and efficiently.

In particular, the following four regularities are observed for the roots of $(\lambda_s) = 0$:

1. *The upper and lower bounds for the first root*

As discussed by Watson (1944), the smallest root λ_s of $J'_{2n}(x) = 0$ can be bounded by

$$\begin{aligned} \sqrt{2n(2n+2)} \quad (\text{for } 1 \leq n \leq 2) \\ \sqrt{2n(2n+3)} \quad (\text{for } n > 2) \end{aligned} < \lambda_s < \sqrt{4n(2n+1)} \quad (92)$$

The first roots can be searched within these upper and lower bounds. Once we obtain the first root, we can generate the subsequent roots efficiently if the following properties are noted.

2. For large λ_s , any two non-zero neighboring roots λ_{s+1} and λ_s of $(x) = 0$ differ by π

First, the following asymptotic form of Bessel function $J_{\omega_n}(x)$ is noted (Watson, 1944)

$$J_{\omega_n}(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\omega_n \pi}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}) \quad (93)$$

For large x , we can retain the first-order term, and consider its differentiation. Thus, we have

$$J'_{\omega_n}(x) \approx -\sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\omega_n \pi}{2} - \frac{\pi}{4}\right) \quad (94)$$

Consequently, the roots of $J'_{\omega_n}(\lambda_s) = 0$ should satisfy approximately

$$\lambda_s - \frac{\omega_n \pi}{2} - \frac{\pi}{4} = k\pi, \quad k = 0, 1, 2, \dots \quad (95)$$

Therefore, the difference between any two neighboring roots is given by

$$\lambda_{s+1} - \lambda_s \approx \pi \quad (96)$$

if λ_s is large.

3. There must be a root of $J'_{\omega_n-1}(x) = 0$ between any two neighboring roots of $J'_{\omega_n}(x) = 0$

To show this property, the following argument is applied. By following the procedure given in Section 15.22 of Watson (1944), we define a function $f(x) = x^{\omega_n+1} J'_{\omega_n}(x)$. Suppose that λ_s and λ_{s+1} are two neighboring roots of $J'_{\omega_n}(x) = 0$, obviously, $f(\lambda_s) = f(\lambda_{s+1}) = 0$. It is well known that there must exist a value ξ between λ_s and λ_{s+1} that satisfies $f'(\xi) = 0$. In addition, it can be proved that $f'(\xi) = \xi^{\omega_n+1} J'_{\omega_n-1}(\xi)$, so we have $J'_{\omega_n-1}(\xi) = 0$ for $\lambda_s < \xi < \lambda_{s+1}$. That is, between any two neighboring roots λ_s and λ_{s+1} of $J'_{\omega_n}(x) = 0$, we can find a root ξ for $J'_{\omega_n-1}(\xi) = 0$.

4. There must be a root of $J'_{\omega_n}(x) = 0$ between any two neighboring roots of $J'_{\omega_n-1}(x) = 0$

The proof for this property is similar to those used in the previous characteristic (3). In particular, we can define a function $g(x) = x^{-\omega_n} + 1 J'_{\omega_n} - 1(x)$, and follow the same procedure employed for the proof of (3). Because the proof follows trivially, the details will not be given here.

By noting these four regularities, the roots of $J'_{\omega_n}(x) = 0$ can be evaluated efficiently.

9. Conclusion

In this paper, we have presented a general solution for a finite isotropic solid circular cylinder subjected to arbitrary boundary loads. Equations of equilibrium are first converted to two uncoupled differential equations by using the displacement function approach. Appropriate solution forms of these displacement functions are proposed in terms of series expression involving Bessel and modified Bessel functions in r -dependency, trigonometric and hyperbolic functions in z -dependency, and trigonometric

functions in θ -dependency. The boundary tractions on the curved surface are expanded into double Fourier series expansion, while those on the end surfaces are expanded into Fourier–Bessel series expansion. System of simultaneous equations for the unknown constants of the displacement functions is given explicitly for any arbitrary boundary loads. It was demonstrated that only one of the end boundary conditions need to be satisfied, while the other end boundary will be satisfied automatically. Solution for axisymmetric problems given by Saito (1952, 1954) for finite solid cylinders, and solutions for the axial and diametral Point Load Strength Test (PLST) given by Wei et al. (1999) and Chau and Wei (1999) can be recovered as special cases of the present general solution. Part II of this study will specialize the present solution to the stress analysis for the double-punch test (Wei and Chau, 1999).

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Appendix A

In order to apply the end boundary condition, the following r -dependent functions $I_{\omega_n}(\eta_m r)$, $\eta_m r I_{\omega_n-1}(\eta_m r)$, $\eta_m^2 r I_{\omega_n}(\eta_m r)$, and $r \frac{\partial J_{\omega_n}(\gamma_s r)}{\partial r}$ can be expressed in series of Bessel function $J_{\omega_n}(\gamma_s r)$ as (Watson, 1944):

$$I_{\omega_n}(\eta_m r) = \sum_{s=1}^{\infty} T_{ms} J_{\omega_n}(\gamma_s r) \quad (\text{A1})$$

$$\eta_m r I_{\omega_n-1}(\eta_m r) = \sum_{s=1}^{\infty} U_{ms} J_{\omega_n}(\gamma_s r) \quad (\text{A2})$$

$$\eta_m^2 r I_{\omega_n}(\eta_m r) = \sum_{s=1}^{\infty} W_{ms} J_{\omega_n}(\gamma_s r) \quad (\text{A3})$$

$$r \frac{\partial J_{\omega_n}(\gamma_s r)}{\partial r} = \sum_{s=1}^{\infty} V_{sn} J_{\omega_n}(\gamma_s r) \quad (\text{A4})$$

where

$$T_{ms} = \frac{2\lambda_s^2}{R^2(\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \int_0^R r I_{\omega_n}(\eta_m r) J_{\omega_n}(\gamma_s r) dr \quad (\text{A5})$$

$$U_{ms} = \frac{2\lambda_s^2 \eta_m}{R^2(\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \int_0^R r^2 I_{\omega_n-1}(\eta_m r) J_{\omega_n}(\gamma_s r) dr \quad (\text{A6})$$

$$W_{ms} = \frac{2\lambda_s^2 \eta_m^2}{R^2 (\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \int_0^R r^3 I_{\omega_n}(\eta_m r) J_{\omega_n}(\gamma_s r) dr \quad (\text{A7})$$

$$V_{sn} = \frac{2\lambda_s^2}{R^2 (\lambda_s^2 - \omega_n^2) J_{\omega_n}^2(\lambda_s)} \int_0^R r^2 \frac{\partial J_{\omega_n}(\gamma_s r)}{\partial r} J_{\omega_n}(\gamma_s r) dr \quad (\text{A8})$$

To integrate Eq. (A5), we first note the following formulas (Watson, 1944):

$$\int_0^R r J_{\omega_n}(\eta_m r) J_{\omega_n}(\gamma_s r) dr = \frac{\eta_m R J_{\omega_n+1}(\eta_m R) J_{\omega_n}(\lambda_s) - \lambda_s J_{\omega_n}(\eta_m R) J_{\omega_n+1}(\lambda_s)}{\eta_m^2 - \lambda_s^2} \quad (\text{A9})$$

and

$$I_{\omega_n}(\eta_m r) = e^{-i\omega_n \pi/2} J_{\omega_n}(i\eta_m r) \quad (\text{A10})$$

where $\lambda_s = \gamma_s R$.

Substitution of Eq. (A10) into Eq. (A9) leads to

$$\int_0^R r I_{\omega_n}(\eta_m r) J_{\omega_n}(\gamma_s r) dr = \frac{\eta_m R I_{\omega_n+1}(\eta_m R) J_{\omega_n}(\lambda_s) + \lambda_s I_{\omega_n}(\eta_m R) J_{\omega_n+1}(\lambda_s)}{\eta_m^2 + \lambda_s^2} \quad (\text{A11})$$

Substitution of Eq. (A11) into (A5) leads to Eq. (74).

We can integrate Eq. (A6) by integration by part, but this procedure is tedious. We propose a simpler approach here. In particular, we first consider the differentiation of the following functions:

$$\begin{aligned} \frac{d}{dr} \left[r^2 I_{\omega_n}(\eta_m r) J_{\omega_n}(\gamma_s r) \right] &= 2(1 - \omega_n) r I_{\omega_n}(\eta_m r) J_{\omega_n}(\gamma_s r) + r^2 \left[\eta_m I_{\omega_n-1}(\eta_m r) J_{\omega_n}(\gamma_s r) \right. \\ &\quad \left. + \gamma_s I_{\omega_n}(\eta_m r) J_{\omega_n-1}(\gamma_s r) \right] \end{aligned} \quad (\text{A12})$$

and

$$\begin{aligned} \frac{d}{dr} \left[r^2 I_{\omega_n-1}(\eta_m r) J_{\omega_n-1}(\gamma_s r) \right] \\ = 2\omega_n r I_{\omega_n-1}(\eta_m r) J_{\omega_n-1}(\gamma_s r) + r^2 \left[\eta_m I_{\omega_n}(\eta_m r) J_{\omega_n-1}(\gamma_s r) - \gamma_s I_{\omega_n-1}(\eta_m r) J_{\omega_n}(\gamma_s r) \right] \end{aligned} \quad (\text{A13})$$

The result given in Eq. (75) can be obtained by virtue of Eq. (A11) together with the equation resulting from subtracting Eq. (A13) $\times \gamma_s$ from Eq. (A12) $\times \eta_m$.

Similarly, for the integration of Eq. (A7) we can consider the differentiation of the following functions:

$$\begin{aligned} \frac{d}{dr} \left[r^3 I_{\omega_n}(\eta_m r) J_{\omega_n+1}(\gamma_s r) \right] \\ = 2r^2 I_{\omega_n}(\eta_m r) J_{\omega_n+1}(\gamma_s r) + \eta_m r^3 I_{\omega_n+1}(\eta_m r) J_{\omega_n+1}(\gamma_s r) + \gamma_s r^3 I_{\omega_n}(\eta_m r) J_{\omega_n}(\gamma_s r) \end{aligned} \quad (\text{A14})$$

and

$$\begin{aligned} & \frac{d}{dr} \left[r^3 I_{\omega_n+1}(\eta_m r) J_{\omega_n}(\gamma_s r) \right] \\ &= 2r^2 I_{\omega_n+1}(\eta_m r) J_{\omega_n}(\gamma_s r) - \gamma_s r^3 I_{\omega_n+1}(\eta_m r) J_{\omega_n+1}(\gamma_s r) + \eta_m r^3 I_{\omega_n}(\eta_m r) J_{\omega_n}(\gamma_s r) \end{aligned} \quad (\text{A15})$$

Integration of the equation resulting from the subtraction of Eq. (A14) $\times \gamma_s$ from Eq. (A15) $\times \eta_m$ leads to

$$\begin{aligned} \int_0^R r^3 I_{\omega_n}(\eta_m r) J_{\omega_n}(\gamma_s r) dr &= R^3 \left[\gamma_s I_{\omega_n}(\eta_m R) J_{\omega_n+1}(\lambda_s) + \eta_m I_{\omega_n+1}(\eta_m R) J_{\omega_n}(\lambda_s) \right] \\ &\quad - 2 \int_0^R \left[\gamma_s r^2 I_{\omega_n}(\eta_m r) J_{\omega_n+1}(\gamma_s r) + \eta_m r^2 I_{\omega_n+1}(\eta_m r) J_{\omega_n}(\gamma_s r) \right] dr \end{aligned} \quad (\text{A16})$$

The integration on the right side of Eq. (A16) can be obtained by replacing ‘ ω_n ’ by ‘ $\omega_n + 1$ ’ in Eqs. (A12) and (A13), and substituting the related terms into Eq. (A16). Finally, Eq. (76) can be obtained by substituting Eq. (A16) into (A7).

The integration for Eq. (A8) can be obtained by using the following procedure. We first note that:

$$r \frac{\partial J_{\omega_n}(\gamma_p r)}{\partial r} = \gamma_p r J_{\omega_n-1}(\gamma_p r) - \omega_n J_{\omega_n}(\gamma_p r) \quad (\text{A17})$$

then Eq. (A8) can be integrated exactly if we know the integration of $r^2 J_{\omega_n} - 1(\gamma_p r) J_{\omega_n}(\gamma_s r)$ and $r J_{\omega_n}(\gamma_p r) J_{\omega_n}(\gamma_s r)$.

When $\gamma_p \neq \gamma_s$, the first integration can be obtained by following the similar procedure as getting the integration in Eq. (A6) simply by replacing the modified Bessel function by the Bessel function. The second integration can be derived directly from Eq. (A9).

When $\gamma_p = \gamma_s$, the integration of $r J_{\omega_n}(\gamma_p r) J_{\omega_n}(\gamma_s r)$ is given by Watson (1944)

$$\int_0^R r J_{\omega_n}(\gamma_s r) J_{\omega_n}(\gamma_s r) dr = \frac{1}{4} R^2 \left[2J_{\omega_n}(\lambda_s) J_{\omega_n}(\lambda_s) - J_{\omega_n-1}(\lambda_s) J_{\omega_n+1}(\lambda_s) - J_{\omega_n+1}(\lambda_s) J_{\omega_n-1}(\lambda_s) \right] \quad (\text{A18})$$

The integration of $r^2 J_{\omega_n} - 1(\gamma_s r) J_{\omega_n}(\gamma_s r)$ can be obtained by integrating the following equation and using Eq. (A18):

$$\frac{d}{dr} \left[r^2 J_{\omega_n}(\gamma_s r) J_{\omega_n}(\gamma_s r) \right] = 2r J_{\omega_n}(\gamma_s r) J_{\omega_n}(\gamma_s r) - 2\omega_n r J_{\omega_n}(\gamma_s r) J_{\omega_n}(\gamma_s r) + 2\gamma_s r^2 J_{\omega_n}(\gamma_s r) J_{\omega_n-1}(\gamma_s r) \quad (\text{A19})$$

This completes the Fourier Bessel expansions given in Eqs. (74)–(78).

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